Effect of Differential Loss on Approximate Solutions to the Coupled Line Equations

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The assumption of zero differential loss between coupling modes in a multimode transmission line reduces the complexity of theoretical analysis. Here we show that in general the approximate solution including differential loss between modes may be computed by convolving the solution for the case of zero differential loss with the Fourier transform of exp $(-|\Delta\alpha u|)$. *The rapidity of loss variation versus frequency is limited to* $(2\Delta\alpha/\Delta\beta)f$ *for transmission lines with high* $\Delta\alpha$.

I. INTRODUCTION

Consider the coupled line equations

$$I_0'(z) = -\Gamma_0 I_0(z) + jc(z)I_1(z) \tag{1}$$

$$I_1'(z) = jc(z)I_0(z) - \Gamma_1 I_1(z). \tag{2}$$

These equations are useful in describing the effects of coupling between a signal mode, represented by a complex wave amplitude I_0, and a single spurious mode, represented by I_1, caused by geometric imperfections in a multimode transmission line. These equations may be derived in two ways from basic principles. The coupled line[1] or generalized telegrapher's equations[2,3] may be derived directly, or the geometric imperfections may be considered discrete; the case of continuous imperfections can then be considered as a limiting form of the discrete case.[4]

Exact solutions for these equations are known in only a few special cases, so considerable attention has been given to approximate solutions.[4,5] A second-order approximate solution is difficult to examine in general; however, Rowe and Warters[4,5] have given a very thorough investigation for the case of equal attenuation for the two modes or zero

differential loss (Re $[\Gamma_0 - \Gamma_1] = 0$). Rowe[6] has shown in the case of a random coupling that the average loss for the TE_{01} in a circular guide mode may be calculated as the convolution of the Fourier transforms of three functions: an attenuation function, a triangular function, and the covariance function of the coupling. Here we show that in general the approximate solution for loss and phase of the TE_{01} mode may be calculated by convolving the solution for zero differential loss with the Fourier transform of the attenuation function.

II. PROOF

Approximate solutions to (1) and (2) are more conveniently described by normalizing the mode amplitudes in the following way

$$G_0(z) = I_0(z)e^{+\Gamma_0 z}$$
$$G_1(z) = I_1(z)e^{+\Gamma_1 z}.$$

The approximate solution[5] for $G_0(z)$ is

$$G_0(z) = 1 - \rho$$

where

$$\rho = \int_0^z e^{\Delta\Gamma u}\, du \int_0^{z-u} c(x)c(x+u)\, dx$$

$$\Delta\Gamma = \Delta\alpha + j\Delta\beta = \Gamma_0 - \Gamma_1$$

and the initial conditions are

$$I_0(0) = G_0(0) = 1$$
$$I_1(0) = G_1(0) = 0.$$

The normalized loss, $A = -\ln |G_0|$, may be further approximated by

$$A = \mathrm{Re}\, \rho$$

and the phase θ by

$$\theta = -\mathrm{Im}\, \rho.$$

2.1 *Loss*

Consider first the case of real coupling $c(z)$; then the normalized loss for a guide of length L is

$$A = \int_0^L e^{\Delta\alpha u} \cos \Delta\beta u\, du \int_0^{L-u} c(x)c(x+u)\, dx.$$

Let

$$\bar{c}(x) = p(x)c(x)$$

where

$$\begin{aligned} p(x) &= 1 \qquad 0 \leqq x \leqq L \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Then we write

$$R(u) = \int_0^{L-u} c(x)c(x+u)\,dx$$

as

$$R(u) = \int_{-\infty}^{\infty} \bar{c}(x)\bar{c}(x+u)\,dx.$$

We observe that $R(u)$ so defined is an even function of u and vanishes for $|u| > L$.

$$R(u) = R(-u)$$

$$R(u) = 0, \qquad |u| > L.$$

Assuming the signal mode is the lowest-loss mode of the transmission line, such as the TE_{01} mode in circular wave-guide, we will have $\Delta\alpha \leqq 0$ and may write the normalized loss as

$$A(\zeta) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-|\Delta\alpha u|}\,R(u)\cos 2\pi\zeta u\,du$$

or

$$A(\zeta) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-|\Delta\alpha u|}\,R(u)\,e^{-j2\pi\zeta u}\,du$$

where

$$2\pi\zeta = \Delta\beta.$$

Thus $A(\zeta)$ is the Fourier transform of the product of two functions and may be written as the convolution of their individual transforms. The transform of the first function is

$$\begin{aligned} B(\zeta) &= \int_{-\infty}^{\infty} e^{-|\Delta\alpha u|}\,e^{-j2\pi\zeta u}\,du \\ &= \frac{2}{|\Delta\alpha|}\,\frac{1}{\left[1 + \left(\frac{2\pi}{\Delta\alpha}\zeta\right)^2\right]}. \end{aligned}$$

The transform of the second function is the solution for $\Delta\alpha = 0$,

$$A_0(\zeta) = \int_{-\infty}^{\infty} \frac{R(u)}{2} e^{-j2\pi\zeta u} \, du$$
$$= \tfrac{1}{2} \mid \bar{C}(\zeta) \mid^2$$

where

$$\bar{C}(\zeta) = \int_{-\infty}^{\infty} \bar{c}(u) \, e^{-j2\pi\zeta u} \, du.$$

Thus

$$A(\zeta) = B(\zeta) * A_0(\zeta)$$

where

$$B(\zeta) * A_0(\zeta) = \int_{-\infty}^{\infty} B(\zeta) A_0(\zeta - \xi) \, d\xi.$$

We observe that for $\Delta\alpha = 0$, $B(\zeta)$ becomes the unit impulse function which is the identity function for the convolution operator

$$A_0(\zeta) = B(\zeta) * A_0(\zeta), \qquad \Delta\alpha = 0.$$

2.2 *Phase*

The same analysis may be applied to the normalized phase as follows

$$\theta(\zeta) = -\int_0^L e^{\Delta\alpha u} \sin \Delta\beta u \, du \int_0^{L-u} c(x)c(x+u) \, dx$$
$$= -\int_0^{\infty} e^{\Delta\alpha u} R(u) \sin \Delta\beta u \, du$$
$$= -\frac{1}{2} \int_{-\infty}^{\infty} e^{-|\Delta\alpha u|} R(u) \operatorname{sgn} u \sin \Delta\beta u \, du$$

where

$$\operatorname{sgn} u = +1, \qquad u > 0$$
$$= -1, \qquad u < 0$$

$$= -\int_{-\infty}^{\infty} e^{-|\Delta\alpha u|} \frac{R(u)}{2} j \operatorname{sgn} u \, e^{-j2\pi\zeta u} \, du$$

then

$$\theta(\zeta) = -B(\zeta) * A_0(\zeta) * \frac{1}{\pi\zeta}$$

or

$$\theta(\zeta) = -A(\zeta) * \frac{1}{\pi\zeta}$$

which means the loss and phase functions are related by Hilbert transforms.[7]

The extension to complex coupling is straightforward and we obtain the following results:

Let

$$\bar{c}(x) = [C_r + jC_i]c(x)$$

then

$$A(\zeta) = B(\zeta) * \left[(C_r^2 - C_i^2)A_0(\zeta) + (2C_rC_i)A_0(\zeta) * \frac{1}{\pi\zeta}\right]$$

$$\theta(\zeta) = -B(\zeta) * \left[(2C_rC_i)A_0(\zeta) + (C_r^2 - C_i^2)A_0(\zeta) * \frac{1}{\pi\zeta}\right]$$

and

$$\theta(\zeta) = -A(\zeta) * \frac{1}{\pi\zeta}$$

since

$$\frac{1}{\pi\zeta} * \frac{1}{\pi\zeta} = \delta(\zeta)$$

where $\delta(\zeta)$ is the unit impulse function.

III. DISCUSSION AND INTERPRETATION OF RESULTS

For simplicity we discuss only real coupling, but the results apply equally well to complex coupling. Consider the representation of $c(x)$ by a Fourier series over the length L with coefficients c_n. Rowe and Warters[4] have shown that for the case of zero differential loss, $A_0(\zeta)$ may be expressed as a double infinite summation

$$A_0(\zeta) = \frac{L^2}{2} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_m c_n^*(-1)^{m-n} \frac{\sin \pi(\zeta L - m) \sin \pi(\zeta L - n)}{\pi(\zeta L - m)\pi(\zeta L - n)}$$

which is an expression for loss in terms of free-space wavelength, since at frequencies considerably above cutoff $2\pi\zeta = D\lambda_0$ (D constant).

$A_0(\zeta)$ is a band-limited function. Its sample points have relative frequency separation $\delta f/f = \delta\zeta/\zeta = 1/2\zeta L$, so the signal mode loss may vary more and more rapidly with frequency as the line length increases,

for $\Delta\alpha = 0$. Since $A_0(\zeta)$ is a band-limited function, $A(\zeta)$ is a band-limited function also. If we consider ζ as a time variable, then $A(\zeta)$ is the result of filtering $A_0(\zeta)$ with a low-pass filter whose impulse response is $B(\zeta)$. If $\Delta\alpha$ is very small $B(\zeta)$ has large amplitude at the origin and approaches zero rapidly away from the origin, and convolving $B(\zeta)$ with $A_0(\zeta)$ results in little change. The most noticeable change is a reduction in the peaks and an increase at the original minimums of the loss fluctuations. However, if $\Delta\alpha$ is increased until the spacing of the half-height points of $B(\zeta)$ is wider than the sample point spacing of $A_0(\zeta)$, the rapidity of the loss fluctuations with frequency will be controlled by $B(\zeta)$. In general for high $\Delta\alpha$ the frequency separation between half height points for $A(\zeta)$ is approximately the same as that of $B(\zeta)$ which is independent of length

$$\frac{\delta f}{f} = \left(\frac{2\Delta\alpha}{\Delta\beta}\right).$$

A numerical example pertaining to the waveguide problem is the following: Consider the previously mentioned loss function for a guide with a two-foot wiggle. The signal mode is TE_{01}. Let

$$L = 1000 \text{ ft.}$$

$$k = 500$$

$$\Delta\alpha = -0.184 \text{ neper/ft.}$$

Now the peak loss point is for $\zeta = 1/2$ which is near 50 kmc for TE_{12} in 2-inch diameter circular copper waveguide. The $\Delta\alpha$ value is typical of TE_{12} in lossy-jacketed helix waveguide. Now consider the half-height points for A_0. This bandwidth is approximately 0.084 kmc, which is very narrow compared to $B(\zeta)$, so that after convolution $A(\zeta) \approx KB(\zeta)$. The half-height points for $B(\zeta)$ are about 6.0 kmc. Thus the addition of the differential loss changes the TE_{01} loss from a very rapidly varying to a very slowly varying function of frequency. This effect is of great value for wideband transmission systems. It is also important in experimental measurements, since the number of measurements necessary for a guide with high loss to the spurious mode is greatly reduced.

Finally we recall that $\bar{c}(x) = p(x)c(x)$ so that

$$\bar{C}(\zeta) = \left(Le^{-j\pi\zeta L}\frac{\sin \pi L\zeta}{\pi L\zeta}\right) * C(\zeta)$$

which makes the solution for a Fourier series representation of $c(x)$ obvious, since $C(\zeta)$ would be a series of impulse functions, and the convolution operation is very easy.

$$A(\zeta) = \tfrac{1}{2}B(\zeta) * \left[\left(Le^{-j\pi\zeta L}\frac{\sin \pi L\zeta}{\pi L\zeta} * C(\zeta)\right)\left(Le^{+j2\pi\zeta L}\frac{\sin \pi L\zeta}{\pi L\zeta} * C^*(\zeta)\right)\right]$$

$$A(\zeta) = B(\zeta) * \left[\frac{L^2}{2}\sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} c_m c_n^*(-1)^{m-n} \cdot \frac{\sin \pi(\zeta L - m)\sin \pi(\zeta L - n)}{\pi(\zeta L - m)\pi(\zeta L - n)}\right].$$

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